Exercise 54

Find the derivative of the function. Simplify where possible.

$$y = \tan^{-1}\left(x - \sqrt{1+x^2}\right)$$

Solution

Use the chain rule and the derivatives of the inverse trigonometric functions listed on page 214.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\tan^{-1}\left(x - \sqrt{1+x^2}\right) \\
&= \frac{1}{1+\left(x-\sqrt{1+x^2}\right)^2}\cdot\frac{d}{dx}\left(x-\sqrt{1+x^2}\right) \\
&= \frac{1}{1+\left[x^2-2x\sqrt{1+x^2}+(1+x^2)\right]}\cdot\left[\frac{d}{dx}(x)-\frac{d}{dx}\sqrt{1+x^2}\right] \\
&= \frac{1}{2+2x^2-2x\sqrt{1+x^2}}\cdot\left[1-\frac{1}{2}(1+x^2)^{-1/2}\cdot\frac{d}{dx}(1+x^2)\right] \\
&= \frac{1}{2\left(1+x^2-x\sqrt{1+x^2}\right)}\cdot\left[1-\frac{1}{2}(1+x^2)^{-1/2}\cdot(2x)\right] \\
&= \frac{1}{2\left(1+x^2-x\sqrt{1+x^2}\right)}\left(1-\frac{x}{\sqrt{1+x^2}}\right) \\
&= \frac{1}{2\left(1+x^2-x\sqrt{1+x^2}\right)}\left(\frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}}\right) \\
&= \frac{\sqrt{1+x^2}-x}{2\left(1+x^2-x\sqrt{1+x^2}\right)\sqrt{1+x^2}} \\
&= \frac{\sqrt{1+x^2}-x}{2\left[(1+x^2)\sqrt{1+x^2}-x(1+x^2)\right]} \\
&= \frac{\sqrt{1+x^2}-x}{2(1+x^2)\left(\sqrt{1+x^2}-x\right)} \\
&= \frac{1}{2(1+x^2)}
\end{aligned}$$